Fluctuation-Dissipation Supplemented by Nonlinearity: A Climate-Dependent Sub-Grid-Scale Parameterization in Low-Order Climate Models

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ABSTRACT

Climate system models use a multitude of parameterization schemes for small-scale processes. These should respond to externally forced climate variability in an appropriate manner so as to reflect the response of the parameterized process to a changing climate. The most attractive route to achieve such a behavior would certainly be provided by theoretical understanding sufficiently deep to enable the à-priori design of climate-sensitive parameterization schemes. An alternative path might, however, be helpful when the parameter tuning involved in the development of a scheme is objective enough so that these parameters can be described as functions of the statistics of the climate system. Provided that the dynamics of the process in question is sufficiently stochastic, and that the external forcing is not too strong, the fluctuation-dissipation theorem (FDT) might be a tool to predict from the statistics of a system (e.g. the atmosphere) how an objectively tuned parameterization should respond to external forcing (e.g. by anomalous sea-surface temperatures). This problem is addressed within the framework of low-order (reduced) models for barotropic flow on the sphere, based on a few optimal basis functions and using an empirical linear sub-grid-scale (SGS) closure. A reduced variant of quasi-Gaussian FDT (rqG-FDT) is used to predict the response of the SGS closure to anomalous local vorticity forcing. At sufficiently weak forcing use of the rqG-FDT is found to systematically improve the agreement between the response of a reduced model and that of a classic spectral code for the solution of the barotropic vorticity equation.

1. Introduction

Both curse, challenge and beauty of atmospheric dynamics is the enormous range of scales involved. Beginning with planetary-scale climate-variability patterns, it extends over synoptic-scale weather and mesoscale systems, such as fronts or gravity waves, down to turbulence on the millimeter scale. Understanding the interactions between these various scales is both a daunting task and a necessity for faithful climate modeling. As has been argued by Held (2005) a hierarchy of models, from full-fledged climate-system models (CSM) down to conceptional models, is needed to gain and keep overview in this complex setting, and thus go on making progress in climate research as a whole. The basis for typical conceptional modeling of atmosphere or ocean dynamics is some kind of filtering. The most classical example is quasigeostrophic theory (Charney 1948), providing the basis for corresponding multi-layer models (Phillips 1954, 1956) as have been applied, e.g., in climate modeling by Opsteegh et al. (1998). Others are soundproof approximations of atmospheric dynamics (Ogura and Phillips 1962; Lipps and Hemler 1982; Lipps 1990; Durran 1989), or the planetary-geostrophic approximation (Robinson and Stommel 1959; Welander 1959; Phillips 1963) which is at the heart of representative earth system models of intermediate complexity (e.g. Petoukhov et al. 2000). An especially compact approach is represented by deterministic low-order models based on some kind of optimal basis patterns (e.g. Selten 1995; Achatz et al. 1995; Kwasniok 1996; Achatz and Schmitz 1997; Selten 1997; Achatz and Branstator 1999; Achatz and Opsteegh 2003a,b; Kwasniok 2004, 2007). Although these have been shown to reproduce various aspects of internal climate variability, they have not yet found their way into practical climate modeling.

Common to both conceptional and full-fledged climate models (even the latter are therefore in some regard conceptional) is that they do not resolve certain small-scale structures or processes (e.g. synoptic-scale or mesoscale systems, clouds etc.) which yet have a non-negligible feedback on the resolved scales. That feedback must be taken into account via suitably formulated sub-grid-scale (SGS) parameterizations. Regardless whether these are given a stochastic (e.g. Hasselmann 1976; Farrell and Ioannou 1993, 1996a,b; Majda et al. 2003; Franzke et al. 2005; Franzke and Majda 2006; Dolaptchiev et al. 2012) or deterministic formulation, nonlinearity and general complexity of the processes in question have so far always prevented a complete à-priori derivation from first principles. The common approach is data driven, i.e. some assumption is made about the functional form of the parameterization, often based on some theory, and the corresponding parameters are obtained in a more or less objective manner by tuning against some reference data set. The crudeness in this procedure varies widely. At one end one might see traditional damping by some hyper-diffusivity, typically tuned via eye-ball comparisons of simulated mean fields or fluxes. A sophisticated approach is stochastic mode reduction suggested by Majda et al. (2003) where the nonlinear self-interaction of unresolved scales is given an empirical description by an Ornstein-Uhlenbeck process which is then used for an explicit derivation of the stochastic SGS parameterization. Somewhat of a middle route is perhaps represented by Achatz and Branstator (1999) who use an empirical linear SGS closure where the parameters have been chosen so as to minimize the mean error between resolved tendencies either predicted by the model or measured in a reference data set, there from an atmospheric general circulation model (GCM).

A perhaps prototypical problem of empirical SGS schemes is confronting Achatz and Branstator (1999): Their low-order models, based on a limited number of empirical orthogonal functions (EOF), simulate the GCM climate very well. Nonetheless they seem to fail to reproduce the climate response of the GCM to some local anomalous thermal forcing. An explanation for this could have been that the nonlinear dynamics of the low-order model, obtained from a projection of the equations of a quasigeostrophic two-layer model onto the EOFs, was too simple. However, analogous attempts by Achatz and Opsteegh (2003a,b), now using primitive equation dynamics, did not solve the problem. Again the GCM climate was simulated well, again the anomalous response to local thermal forcing could not be reproduced to a satisfactory degree. Still a possible explanation could be that the dynamics of the low-order model, using the dry primitive equations on three layers, is too far away from that of the much more sophisticated 19-level GCM (described by Voss et al. 1998). The problem might, however, be deeper and more general than that: as the SGS-scheme parameters have been determined by tuning against the unperturbed GCM climate, and as that works so well, it might be that an SGS scheme tuned à-posteriori against the perturbed climate could enable the low-order model to reproduce the anomalous response. In other words, the SGS closure should be formulated climate dependent. This is perhaps a problem to be faced by many SGS parameterizations in climate models: The less they are based on first principles, and the more they rely on tuning against present-day or past climate, the more they might be in danger of failing in a changing climate.

The ideal approach to tackle this problem would be the development of SGS schemes based sufficiently on first principles so that the empirical parameters do not matter that much anymore. Perhaps stochastic mode reduction (Majda et al. 2003; Dolaptchiev et al. 2012) points into a direction helping under some circumstances, as is also suggested by Majda et al. (2010) who show a reduced stochastic model of a three-component system to exhibit a realistic response to external perturbations. One might also reconsider the tuning processes for the SGS parameterization. The minimization of relative entropy between the statistics of low-order model and GCM might lead to reduced models with a more faithful climate sensitivity (Majda and Gershgorin 2010, 2011a,b; Branicki and Majda 2012). However, we here follow another route. As long as the parameter tuning implies the minimization of some objectively formulated error, e.g. in predicted tendencies, between model and reference data set, a reasonable à-priori estimate of the change in the corresponding statistics could help. Fortunately, under certain conditions such an estimate can be obtained from the fluctuation-dissipation theorem (Deker and Haake 1975; Hänggi and Thomas 1977; Risken 1984; Gritsun 2001; Gritsun et al. 2002; Gritsun and Branstator 2007; Abramov and Majda 2009; Majda et al. 2010; Cooper and Haynes 2011). For an analysis of the potential of this approach we have restrained ourselves to a minimal framework we hoped to contain all necessary ingredients. Instead of a full-fledged GCM, or even real climate data, we use a spectral code for the barotropic vorticity equation on the sphere as toy atmosphere, construct a low-order model based on EOFs (Selten 1995), and use the empirical SGS parameterization as proposed by Achatz and Branstator (1999). Our results indicate that the fluctuation dissipation theorem (FDT) is not only able to improve the performance of the low-order model in simulating the response to anomalous vorticity forcing, but that the corresponding prediction is also better than that from the most frequently used quasi-Gaussian variant of the FDT itself.

The manuscript is structured as follows: Section 2 describes the toy atmosphere, the approach for construction of a low-order model for its simulation, and some characteristics of the latter. Section 3 gives an account of how we use the FDT for formulating the climate dependence of the SGS closure of the low-order model, while section 4 presents results on how well this approach works for the simulation of the response to anomalous vorticity forcing. Finally we summarize and discuss our findings in section 5.

2. Toy atmosphere and low-order climate model

a. Toy atmosphere

The toy atmosphere used here is a spectral code (Selten 1995) for the solution of the barotropic vorticity equation

$$\frac{\partial \nabla^2 \psi}{\partial t} + J\left(\psi, \nabla^2\psi + f + f_0\frac{h}{H}\right) = -k_E\nabla^2\psi - k_h\nabla^6\psi + F \tag{1}$$

on the sphere. Here ψ is the streamfunction, J the standard Jacobian operator, f the Coriolis parameter, f_0 a midlatitude value of the latter (at 45°N), h/H a normalized envelope orography, k_E represents Ekman damping (with a time scale of 15d), k_h is the hyper-diffusion coefficient (damping the shortest total wavelengths with a time scale of 3d), and F is a forcing tuned by Franzke et al. (2005) so as to lead to a model variability as representative of available northern-hemisphere analysis data as possible. The spherical-harmonic expansion of the streamfunction is truncated in a triangular manner at T21. Since the model is constrained to be symmetric with respect to the equator it has $N = 231$ degrees of freedom, i.e. non-zero vorticity spectral coefficients where real and imaginary parts count separately. Gathering these in a state vector $\mathbf{x} \,\epsilon\, \mathbb{R}^N$, the dynamical equation of our toy atmosphere can be written

$$\frac{d\mathbf{x}}{dt} = \mathbf{G}\left(\mathbf{x}\right) \tag{2}$$

where $\mathbf{G}$ is the appropriate function.

b. Low-order climate model

Instead of spherical harmonics our low-order climate model uses as basis functions empirical orthogonal functions (EOF). These have been extracted from data from 200000d of our toy atmosphere. An energy metric has been employed (Selten 1995) for this so that the norm

$$\begin{aligned} |\mathbf{x}|^2 &= a^2 \int_0^{2\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\phi \cos\phi \, |\nabla\psi|^2 \\ &= \sum_{m=1}^{21} \sum_{n=m}^{21} n\left(n+1\right) |\psi_{mn}|^2 = \mathbf{x}^t \mathsf{M} \mathbf{x} \end{aligned} \tag{3}$$

is proportional to the total energy of the flow, where a is the radius of the earth, λ and ϕ geographic longitude and latitude, and ψ_{mn} a spectral coefficient at zonal and total wavenumbers m and n. The corresponding metric M is an $N \times N$-dimensional real symmetric matrix. We found that 43 EOFs suffice to explain more than 90% of the variance in the analyzed data. In general, if M leading EOFs are chosen to approximate the state vector, the latter can be written

$$\mathbf{x} = \langle\mathbf{x}\rangle + \mathsf{P}\mathbf{a} + \varepsilon \tag{4}$$

where angle brackets indicate the time mean, P is an $N \times M$-matrix containing the EOFs as columns, $\mathbf{a} \,\epsilon\, \mathbb{R}^M$ is the vector of EOF expansion coefficients (the principal components), and ε the time dependent truncation error. The latter is orthogonal to the EOFs so that the principal components can be determined from the data via

$$\mathbf{a} = \mathbf{\mathsf{P}}^t \mathbf{\mathsf{M}} \mathbf{x}' \tag{5}$$

where a prime indicates deviations from the mean, i.e. here $\mathbf{x}' = \mathbf{x} - \langle \mathbf{x} \rangle$.

Taking the time derivative, and using the dynamical equation (2) of the toy atmosphere together with the reduced representation (4) one gets

$$\frac{d\mathbf{a}}{dt} = \mathbf{\mathsf{P}}^t \mathbf{\mathsf{M}} \mathbf{G} \left(\langle \mathbf{x} \rangle + \mathbf{\mathsf{P}} \mathbf{a} \right) + \mathbf{s} \left(\mathbf{x}, \mathbf{a} \right) \tag{6}$$

where $\mathbf{s}$ is the SGS error arising from the neglect of the truncation error inside $\mathbf{G}$. A low-order climate model for $\mathbf{a}$ is obtained by replacing the SGS error by a suitably chosen parameterization $\mathbf{p}\left(\mathbf{a}\right)$. The model equations are then

$$\left(\frac{d\mathbf{a}}{dt} \right)_M = \mathbf{P}\left(\mathbf{a}\right) + \mathbf{p}\left(\mathbf{a}\right) \qquad , \tag{7}$$

the shortcut $\mathbf{P}\left(\mathbf{a}\right) = \mathbf{\mathsf{P}}^t \mathbf{\mathsf{M}} \mathbf{G} \left(\langle \mathbf{x} \rangle + \mathbf{\mathsf{P}} \mathbf{a} \right)$ indicating the projected model without SGS parameterization. Note that the (toy) atmosphere data do not satisfy (7) but rather

$$\frac{d\mathbf{a}}{dt} = \mathbf{P}\left(\mathbf{a}\right) + \mathbf{p}\left(\mathbf{a}\right) + \varepsilon_p \left(\mathbf{x}, \mathbf{a} \right) \qquad , \tag{8}$$

with a parameterization error $\varepsilon_p \left(\mathbf{x}, \mathbf{a} \right) = \mathbf{s}\left(\mathbf{x}, \mathbf{a} \right) - \mathbf{p}\left(\mathbf{a}\right)$. Following Achatz and Branstator (1999) we now choose a linear parameterization

$$\mathbf{p}\left(\mathbf{a}\right) = \mathbf{F} + \mathbf{\mathsf{L}} \mathbf{a} \tag{9}$$

and, instead of tuning the vector $\mathbf{F}$ and the matrix $\mathbf{\mathsf{L}}$ by test integrations and eyeball fits of the climate-model climatology, we determine them by the requirement that, averaged over the available data, the norm of the parameterization error is to be as small as possible. This amounts to the solution of a linear regression problem, yielding

$$\mathbf{\mathsf{L}} = \langle \mathbf{s}' \mathbf{a}'^t \rangle \langle \mathbf{a}' \mathbf{a}'^t \rangle^{-1} \tag{10}$$

$$\mathbf{F} = \langle \mathbf{s} \rangle - \mathbf{\mathsf{L}} \langle \mathbf{a} \rangle \qquad . \tag{11}$$

This way the SGS-closure parameters in (9) are determined from the climate statistics of the toy atmosphere. Necessary input are the covariance $\langle \mathbf{s}'\mathbf{a}'^t \rangle$ between SGS error $\mathbf{s}$ and the state vector $\mathbf{a}$ modeled by the climate model, the auto-covariance $\langle \mathbf{a}'\mathbf{a}'^t \rangle$ of the latter, its mean $\langle \mathbf{a} \rangle$, and the mean SGS error $\langle \mathbf{s} \rangle$. Fig. 1 shows mean streamfunction and streamfunction variance from daily data from 200000d of the toy atmosphere, and the corresponding results from low-order models based on 40 EOFs, either without or with SGS parameterization. The improvement achieved by the parameterization is evident.

3. Climate dependent SGS closure by the fluctuation-dissipation theorem

a. External forcing

The question now is whether the low-order climate model can respond correctly to some external atmospheric forcing. As in Achatz and Branstator (1999) and Achatz and Opsteegh (2003b) we choose a local forcing. However, since the variability of the toy atmosphere is low close to the equator, and thus the leading EOFs would not be well able to represent a tropical forcing, we have rather chosen to place it at midlatitudes. The vorticity forcing is of the form

$$\delta F_\zeta = A \cdot 5 \cdot 10^{-6} f \cos^2\left(\frac{\lambda - \lambda_c}{\Delta\lambda}\right) \cos^2\left(\frac{\phi - \phi_c}{\Delta\phi}\right) \qquad (12)$$

The scaling has been chosen so that the anomalous forcing is at $A = 1$ of the same order as the climatological forcing, i.e. $\delta F_\zeta / F = O(1)$. In all experiments to be discussed here the forcing is centered at latitude $\phi_c = 45°$, its width is $\Delta\lambda = \Delta\phi = 20°$, and it has amplitude $A = 0.1$. In total we will base our conclusions on experiments with center longitude of the forcing being at $\lambda_c = 0°, 30°, \ldots, 330°$. It has always been projected on the same EOFs as the corresponding low-order climate models are using. As an example we show in Fig. 2 the case $\lambda_c = 180°$, either total, projected onto 40 EOFs, or projected onto 90 EOFs.

b. *Fluctuation-dissipation theorem*

The external forcing changes the statistics of the toy atmosphere so that a low-order climate model should incorporate this effect in predicting the atmosphere response to the forcing. In principle, one can always do the experiment and obtain a perturbed closure à-posteriori from data from the perturbed climate, using (10) and (11). The perturbed model would then be

$$\frac{d\mathbf{a}}{dt} = \mathbf{P}\left(\mathbf{a}\right) + \mathbf{F} + \mathsf{L}\mathbf{a} + \delta\mathbf{F} + \delta\mathsf{L}\mathbf{a} + \mathsf{P}^t\mathsf{M}\delta\mathbf{F}_\zeta \tag{13}$$

where $\mathsf{P}^t\mathsf{M}\delta\mathbf{F}_\zeta$ represents the anomalous-forcing spectral coefficients projected onto the EOFs, and

$$\delta\mathsf{L} = \delta\left(\langle\mathbf{s}'\mathbf{a}'^t\rangle\langle\mathbf{a}'\mathbf{a}'^t\rangle^{-1}\right) \tag{14}$$

$$\delta\mathbf{F} = \delta\langle\mathbf{s}\rangle - \delta\left(\mathsf{L}\langle\mathbf{a}\rangle\right) \tag{15}$$

are the corrections in the closure due to the changing climate. Obviously this à-posteriori tuning would make the low-order model useless. Only if the changing statistics can be predicted before-hand this would be a viable option.

Fortunately, the fluctuation-dissipation theorem (Kraichnan 1959; Risken 1984) offers a way how this prediction could be done approximately, albeit under certain assumptions. It considers either a deterministic system governed by

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\left(\mathbf{x}, t\right) \tag{16}$$

or a stochastic system controlled by the corresponding stochastic differential equation

$$d\mathbf{x} = \mathbf{A}\left(\mathbf{x}, t\right)dt + \mathsf{B}\left(\mathbf{x}, t\right)d\mathbf{W} \tag{17}$$

where $\mathbf{x}$ is the state vector of the system, here of the (toy) atmosphere, $\mathbf{A}$ is the deterministic drift, B the diffusion tensor, and $d\mathbf{W}$ a multidimensional Wiener process. The applicability of the FDT to deterministic systems is often hampered by the fractality of the corresponding probability-density function (PDF). In such cases it can help to add a small noise term, as in

(17), to ensure that the PDF is sufficiently smooth (Zeeman 1988). In our application here, e.g., the underlying system is not stochastic, but we assume that the nonlinear dynamics of the smallest-scale processes acts in a sufficiently irregular manner so that stochasticity is a reasonable approximation. To proceed, the general FDT predicts the response of the statistics of the system to an *infinitesimally small perturbation* $\delta\mathbf{f}(\mathbf{x},t)$ of the drift vector so that

$$\mathbf{A}(\mathbf{x},t) \to \mathbf{A}(\mathbf{x},t) + \delta\mathbf{f}(\mathbf{x},t) \qquad . \tag{18}$$

It provides an estimate of the change in the expectation of any observable $\mathbf{h}(\mathbf{x})$, i.e. of

$$\langle\mathbf{h}\rangle(t) = \int d^N x\, p(\mathbf{x},t)\, \mathbf{h}(\mathbf{x}) \tag{19}$$

where p is the PDF. This is also the situation encountered in our problem. Due to (5) and $\mathbf{s} = \mathbf{s}(\mathbf{x},\mathbf{a})$ all the climate means needed for the determination of the SGS closure parameters in (11) and (10) are expectations of suitably defined observables $\mathbf{h}(\mathbf{x})$. Quasi-Gaussian FDT (qG-FDT), the most frequently used variant of FDT assumes that the equilibrium PDF is Gaussian. This is is a certain restriction. Cooper and Haynes (2011) suggest how to relax it by estimating the equilibrium PDF by a kernel method. An alternative, not inherently restricted to low-dimensional applications as there, is the blended short-time/quasi-Gaussian FDT method (ST/qG-FDT) developed by Abramov and Majda (2009). This approach, superior to qG-FDT, uses a tangent linear model to determine the short-time response to external forcing, combined with qG-FDT for longer response times. For the time being, however, we want to stick with qG-FDT, since it is more easily implemented than ST/qG-FDT and since it typically requires considerably less reference data than kernel methods. Under the assumption of Gaussianity the predicted *steady-state response* for $t \to \infty$ to an *anomalous forcing*

$$\delta\mathbf{f}(\mathbf{x},t) = \delta\mathbf{f}(t) \tag{20}$$

is

$$\lim_{t\to\infty} \delta\langle\mathbf{h}\rangle(t) = \lim_{t\to\infty} \int_0^t d\tau \langle\mathbf{h}[\mathbf{X}(\tau)]\,\mathbf{X}'^t(0)\rangle\langle\mathbf{x}'\mathbf{x}'^t\rangle^{-1}\delta\mathbf{f}(t-\tau) \tag{21}$$

which yields for constant forcing, as here,

$$\lim_{t\to\infty} \delta\langle \mathbf{h}\rangle (t) = \mathsf{R}\delta\mathbf{f} \qquad , \tag{22}$$

with qG-FDT response operator

$$\mathsf{R} = \int_0^\infty d\tau \langle \mathbf{h}\left[\mathbf{X}(\tau)\right]\mathbf{X}'^t(0)\rangle\langle \mathbf{x}'\mathbf{x}'^t\rangle^{-1} \tag{23}$$

The task is to determine the integral over all time lags of the lagged covariance between the observable and the state vector, multiplied by the inverse of the lag-zero auto-covariance matrix. This offers a way for the determination of the response of all SGS closure parameters in (11) and (10), or rather of the expectations $\langle \mathbf{a}\rangle$, $\langle \mathbf{s}\rangle$, $\langle \mathbf{s}'\mathbf{a}'^t\rangle$, and $\langle \mathbf{a}'\mathbf{a}'^t\rangle$ required for their calculation.

c. Reduced quasi-Gaussian FDT

Referring to (10) and (11) we note again that the observables we need modified means for are the reduced state vector $\mathbf{a}$, the SGS error $\mathbf{s}$, the matrix $\mathbf{s}'\mathbf{a}'^t$ yielding in the mean the covariance of the SGS error with the reduced state vector, and the matrix $\mathbf{a}'\mathbf{a}'^t$ averaging to the auto-covariance. The ability of qG-FDT to predict the atmospheric response in these observables has been estimated by performing 12 experiments with a mid-latitude anomalous forcing with amplitude $A = 0.1$ at longitudes $\lambda_c = 0°, 30°, \ldots, 330°$, and projected onto the leading 40 EOFs. Each case has been integrated for 200000d. A reference case has been obtained by integrating the model over 500000d, and the qG-FDT response operator has been approximated by integrating the lagged covariances in the reference data over 50d. For this a simple Riemann sum has been used, with a time step of 1d. Following Gritsun and Branstator (2007) and Majda et al. (2010) we have not determined the operator in the full state space but rather in the state space spanned by the leading 40 EOFs. The estimate of the response in the four quantities in question has then been obtained using qG-FDT, and that has been compared to the true response from the toy atmosphere. The comparison has been made by calculating a relative error in EOF space, defined, either for two vectors $\mathbf{a}$ and $\mathbf{b}$, or for two matrices A and B, as

$$\epsilon = \begin{cases} \dfrac{|\mathbf{a}-\mathbf{b}|^2}{|\mathbf{a}|\,|\mathbf{b}|} \\ \dfrac{|\mathsf{A}-\mathsf{B}|^2}{|\mathsf{A}|\,|\mathsf{B}|} \end{cases} \tag{24}$$

where the norm of a matrix is taken to be the Frobenius norm, i.e. the square root of the squared sum of matrix elements. Pattern correlations, not shown here, have been calculated as well, without yielding any further insights.

The results are shown in Fig. 3. With the one exception of the forcing located at $\lambda_c = 270°$, the anomalous first moments $\langle\mathbf{a}\rangle$ and $\langle\mathbf{s}\rangle$ are predicted by qG-FDT with an error less than 1, whereas the second moments $\langle\mathbf{s}'\mathbf{a}'^t\rangle$ and $\langle\mathbf{a}'\mathbf{a}'^t\rangle$ are not predicted so well. This implies that the prediction of the change in the linear operator of the SGS closure, using (14), could be flawed. As Fig. 4 shows, this is indeed the case. The error between the change either estimated from qG-FDT or obtained à-posteriori from the data of the perturbed atmosphere is always of order 1 or larger. The same holds for the prediction of the change of the forcing of the SGS parameterization, using (15), since it uses the ill-estimated $\delta\mathsf{L}$.

It turns out, however, that if $\delta\mathsf{L}$ is neglected, a useful estimate of $\delta\mathbf{F}$ can be obtained. This *reduced quasi-Gaussian FDT (rqG-FDT)* uses

$$\delta\mathsf{L} = 0 \tag{25}$$

$$\delta\mathbf{F} = \delta\langle\mathbf{s}\rangle - \mathsf{L}\delta\langle\mathbf{a}\rangle \quad , \tag{26}$$

Its quality is shown in Fig.4 as well. Cases of anomalous vorticity forcings projected on more or less EOFs (20, 30, ..., 90), and low-order models of corresponding resolution have been investigated as well, indicating a general potential of reduced qG-FDT to make a useful prediction of the anomalous SGS forcing. The 90-EOF case, e.g., investigated below in somewhat greater detail, shows the same qualitative results as the 40-EOF case discussed here. Therefore our method of choice applied below is rqG-FDT.

4. Results

The utility of reduced qG-FDT is eventually decided by its ability to help a low-order climate model in simulating the atmospheric response to anomalous vorticity forcing. Our respective results will first be illustrated using a representative example. This is the case of anomalous forcing at longitude $\lambda_c = 210°$ projected onto 90 EOFs. Fig. 5 shows the mean-streamfunction response of the toy atmosphere to this forcing, the corresponding response from three different 90-EOF models, and that predicted by qG-FDT. The low-order climate model CM0 uses an unmodified SGS parameterization, the model CMP applies a parameterization modified à-posteriori by a new determination of the SGS-model parameters from the perturbed atmosphere, and the model CMF uses a parameterization modified before-hand via reduced qG-FDT. This is also compared to the direct prediction of the streamfunction response by qG-FDT. The atmospheric response has a strong zonal component with maxima over the pole and in the subtropics, and a minimum in midlatitudes. A wave component exhibits a maximum over the subtropical Pacific, and three minima over the midlatitude Pacific and Atlantic ocean and over Siberia. This pattern is reproduced quite well even by CM0. The relative error in the simulated response is 0.124. CMF brings an improvement, e.g. over the Pacific, so that the relative error drops to 0.058. This is even better than the direct qG-FDT result which has a relative error of 0.228. For better orientation Fig. 6 shows the change in the mean zonal wind, exhibiting an intensification and eastward shift of the two jet streams. Maximum values are about 9m/s. This is of the same order as zonal-mean zonal wind changes in present-day simulations of anthropogenic climate change (e.g. Lorenz and DeWeaver 2007). Notwithstanding its linear nature rqG-FDT is able to predict the change in the SGS parameterization well enough that model CMF can simulate that mean zonal-wind change faithfully.

As discussed in subsection 3c, standard qG-FDT is well able to predict the change in first moments of the atmosphere. This is also visible in the results shown so far. Second moments, however, had been shown to be more difficult an object for qG-FDT. This is born out in Fig.

7, where the response in the streamfunction variance is shown. qG-FDT predicts a signal which is considerably too strong (relative error 2.47). Here the nonlinear models perform better, especially if supplemented by a rqG-FDT modification of the SGS parameterization. The predicted response is slightly too weak, but the predicted pattern is matched very well, with an increase of variance over the pole, and north and south of the jet streams. The relative errors are 0.527 for CM0 and 0.342 for CMF.

Relative errors, yielding a quantitative estimate of the quality of the simulated response, have not just been calculated for an anomalous forcing at longitude $\lambda_c = 210°$, but for all twelve cases examined. Fig. 8 shows the relative errors in the predictions of the change in the first moments. With the exception of the three cases with forcing longitude between 210° and 270°, qG-FDT is better able to predict the response than the unmodified climate model. Only in four out of the twelve cases ($\lambda_c = 60°, 180°, 300°$ and $330°$), however, rqG-FDT is not able to improve the climate model so much that it can outdo qG-FDT. The balance in favor of climate models supplemented by rqG-FDT becomes even more convincing in the case of the second moments, shown in Fig. 9. Here it is always the climate model using rqG-FDT for the adjustment of the SGS parameterization that gives the better prediction. Note also that the model with SGS parameterization modified à-posteriori is always performing best. Although this model is useless in itself, this fact demonstrates that there might be even more potential in the approach, should it become possible to also predict the second-moment change better than qG-FDT is able to.

Finally, we give an overview how our results depend on the number of EOFs which the anomalous forcing and the climate models are based on. This is to give an indication on how well the approach might work at various conceivable levels of climate-model simplicity in comparison with the true complexity of the atmosphere. For this purpose, we have calculated, for either the first- or second-moment errors, a mean over all twelve cases and a root-mean-square deviation. Mean plus/minus r.m.s. deviation of the first-moment errors are shown in Fig. 10 for models based on between 20 and 90 EOFs. The smallest models, based on only 20

EOFs, are performing worse than the qG-FDT, even if the SGS parameterization is adjusted à-posteriori. At model resolutions too coarse the linear ansatz for the SGS parameterization cannot compete with qG-FDT. This already changes, however, at a resolution of 30 EOFs. At this and all higher resolutions simulations by an optimally adjusted nonlinear climate model can outperform direct application of qG-FDT. Nonetheless, application of rqG-FDT helps to improve the model behavior at all examined resolutions. Models based on 60 or 70 EOFs become as good as qG-FDT if rqG-FDT is used to adjust the parameterization, and at higher resolutions they perform better. This gain becomes even more obvious as one looks at the second moments. Fig. 11 shows the weakness of qG-FDT in predicting the anomaly in these, but also that the climate-model simulations can yield useful results, especially for models with higher resolution and supplemented by rqG-FDT. The modification of the SGS parameterization by rqG-FDT gives an approximate net 30% improvement over models without modified parameterization, and considerably more over the direct application of qG-FDT, when 80 or 90 EOFs are used.

5. Summary and discussion

We have addressed the question how sub-grid-scale (SGS) parameterizations in climate models can be formulated so that they respond correctly to an externally forced change in climate statistics. For this purpose we have considered a toy atmosphere represented by a spectral code, with resolution T21, for the solution of the barotropic vorticity equation on the sphere. The vorticity forcing in that code has been chosen so that its climate exhibits a certain similarity to that of the real atmosphere. Low-order climate models have then been constructed which are based on empirical orthogonal functions (EOF). For this an energy metric has been used. The identified variance spectrum is relatively flat. About 40 basis patterns are needed for representing 90% of the total variance of the toy atmosphere. The dynamical equations of the climate models, varying by the number of EOFs they are based on, have been obtained by projecting the T21 code for the barotropic vorticity equation onto the EOFs, and by adding an SGS parameterization which is to describe the feedback from unresolved modes. That parameterization has been given a formulation which is linear in terms of the EOF expansion coefficients. The respective parameters, comprised in an SGS forcing and a linear SGS operator, have been determined from the toy-atmosphere climate in such a manner that the residual error between modeled and measured tendencies, averaged over all available climate data, is as small as possible. This represents an objective tuning process.

Parameters of an SGS parameterization tuned at present-day climate might have to respond to climate change. We suggest that the fluctuation-dissipation theorem (FDT) is used to predict this response. Corresponding response operators have been constructed from the toy-atmosphere climate data, assuming their probability-density function (PDF) to be Gaussian (qG-FDT). This is a limiting assumption which one could potentially relax. A more general treatment would, however, either necessitate the estimate of the PDF by kernel methods (Cooper and Haynes 2011) or require the use of a tangent-linear model for the determination of the short-time response to external forcing (Abramov and Majda 2009). Kernel methods can become computationally expensive, and they are inherently restricted to low-dimensional applications. The ST/qG-FDT method of Abramov and Majda (2009) does not suffer from this problem. It might be an option to be tested in the future. However, we also speculate that, due to the central-limit theorem, the deviations from Gaussianity might become the smaller the more complex, and thus realistic, the examined setting becomes. The ability of the qG-FDT to predict the response of the SGS parameterizations has been investigated using the example of anomalous local vorticity forcings in midlatitudes, at twelve different equidistant positions in geographic longitude, and projected on the EOF bases which the corresponding low-order models use. It is found that qG-FDT can predict the response in the first moments of the toy-atmosphere climate well, not however that of the second moments. This is in line with the findings of Majda et al. (2005) that, for systems with quadratic nonlinearity, use of a Gaussian PDF in (21) yields third-order accurate results for the first moments, while those for the second moments are second-order accurate. Indeed Gritsun et al. (2008) found a worse, albeit reasonable, qG-FDT performance for second than for first moments. Moreover, the forcing chosen here is sub-optimal, as Abramov and Majda (2009) show that ST/qG-FDT applied to this system works best for anomalous forcings projecting onto the leading EOF 1. This is no real surprise since the toy-atmosphere with its 231 degrees of freedom is only roughly consistent with the basic assumptions of the FDT. Only barotropic Rossby waves are present. Neither does the toy atmosphere allow comparatively fast synoptic-scale processes such as baroclinic instability, nor does it contain gravity waves. Thus a basic picture of slow modes stochastically forced by components with much shorter intrinsic time scales is not met very well so that the system PDF could have a stronger fractality than allowed under ideal conditions. Corresponding extensions should be considered in the future. In the present context, however, the reliable qG-FDT prediction of the changes in the first moments can be used to predict the response of the SGS forcing to the external forcing. In an approach which we call reduced qG-FDT (rqG-FDT) this has been done while the linear SGS operator has been kept untouched.

The reduced qG-FDT has then been applied to the various anomalous-forcing cases. Low-order models with SGS closure adjusted via rqG-FDT have been investigated for their ability in predicting first- and second-moment anomalies in the data of the perturbed toy atmosphere. This has been compared to the potential of low-order models without adjusted parameterization, or of the direct application of qG-FDT. Only very small models, based on only 20 EOFs, perform worse than qG-FDT. With a basis of intermediate size (30 – 60 EOFs) they are more successful in predicting the second-moment anomalies, i.e. the anomalous fluxes, while direct application of qG-FDT. gives a more reliable prediction of the first-moment anomalies, i.e. the anomalous streamfunction. There, however, rqG-FDT is already able to improve the low-order model prediction, as compared to simulations without modified SGS parameterization. An encouraging result is that models based on sufficiently many EOFs (80 or 90) perform clearly best if adjusted via rqG-FDT. Both first- and second-moment anomalies are simulated better (by about 30% for second moments) than by models without adjusted parameterization, or (by even more) than by the direct application of the qG-FDT.

It thus seems that the combination of FDT and explicit simulations of the nonlinear system is a promising approach which should be pursued further. Another promising route specifically towards realistic low-order modelling of the atmospheric climate could perhaps be the combination of a reduced stochastic model with non-Gaussian FDT. At least for a three-component system Majda et al. (2010) show this approach to be significantly more powerful than the application of qG-FDT to the complete system. One might also reconsider the tuning process for the SGS parameterization by using concepts from information theory (Majda and Gershgorin 2010, 2011a,b; Branicki and Majda 2012). However, given the encouraging results we have here we see our comparatively simple approach as supplementary to such ideas. Interesting here is also that low-order models with SGS parameterization determined directly from anomalous data of the toy atmosphere were performing even better than all other models. This indicates that more general FDT approaches not relying on quasi-Gaussianity (Abramov and Majda 2009; Cooper and Haynes 2011) could yield better results. It also indicates, however that as soon as the FDT as a whole is better able to also predict the second-moment anomalies, its application to the adjustment of the SGS parameterization might lead to even more useful results. We hope that this will be borne out in the future when the approach will have been applied to toy atmospheres, or even real climate data, with a better time-scale separation between slow and fast processes as here. Baroclinic models, allowing for baroclinic instability, or unbalanced models, containing gravity waves, will be interesting testbeds to be examined. As a corresponding encouragement we see the findings of Gritsun et al. (2008) where in an application of qG-FDT to GCM data even the second-moment results were quite reasonable.

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List of Figures

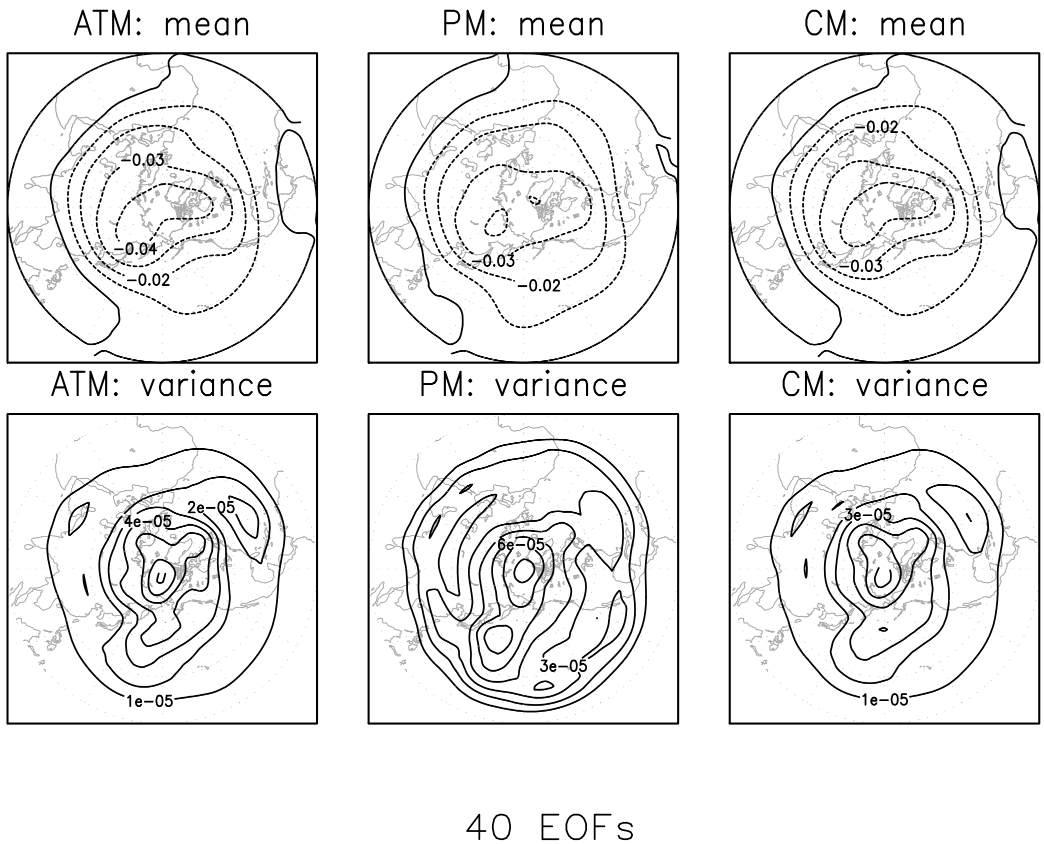

FIG. 1. Mean streamfunction (top row, contour interval 0.01) and streamfunction variance (bottom, contour interval $1 \cdot 10^{-5}$), from 200000d of data from the toy atmosphere (left column), a projected 40-EOF model without SGS parameterization (middle), and a 40-EOF climate model with SGS parameterization (right). The streamfunction has been normalized by $a^2\Omega$ with Ω the angular frequency of the earth.

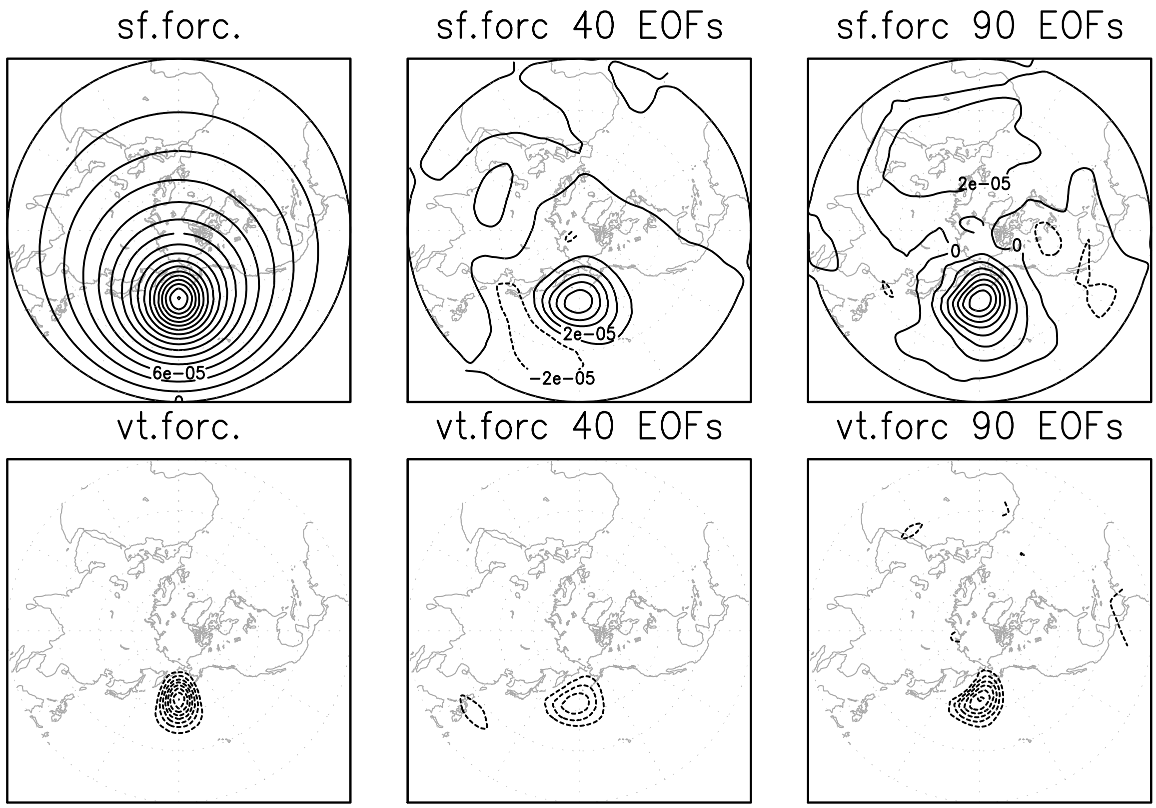

FIG. 2. Vorticity forcing (bottom row, contour interval $1 \cdot 10^{-3}$, only negative contours shown) and corresponding streamfunction forcing (top, contour interval $2 \cdot 10^{-5}$), centered at $(\lambda_c, \phi_c) = (180°, 45°)$, and nondimenzionalized by length scale a and time scale Ω^{-1}. Shown are the total forcing (left column), the results one obtains from projection onto the leading 40 EOFs (middle), and the result for 90 EOFs (right).

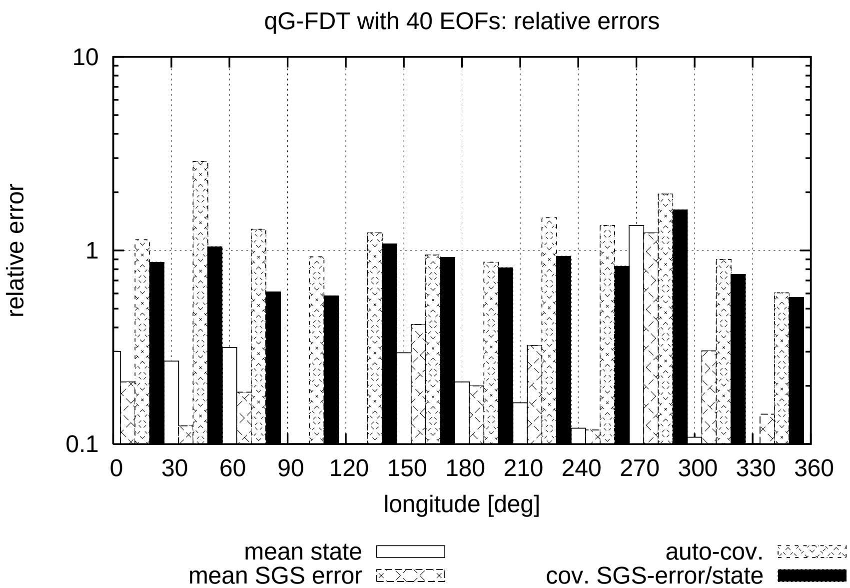

FIG. 3. Relative error in using quasi-Gaussian FDT (qG-FDT) for predicting the response of the toy atmosphere to anomalous local forcing at twelve different longitudes, and projected onto the leading 40 EOFs. Errors have been calculated for the response in the mean reduced state $\langle \mathbf{a} \rangle$, the mean SGS error $\langle \mathbf{s} \rangle$, the covariance of the SGS error with the reduced state vector $\langle \mathbf{s}'\mathbf{a}'^t \rangle$, and the reduced auto-covariance $\langle \mathbf{a}'\mathbf{a}'^t \rangle$. The qG-FDT operator is based on 40 EOFs as well.

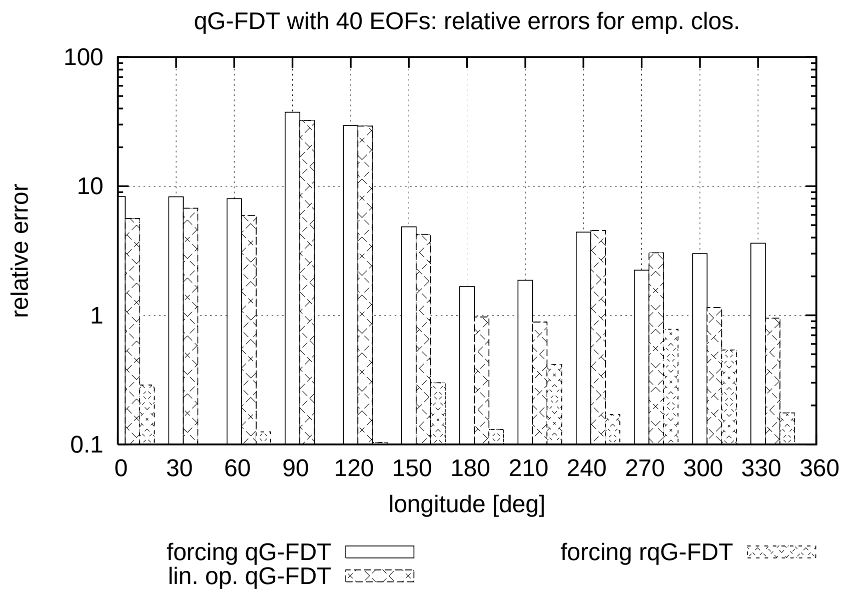

FIG. 4. For the same twelve cases as discussed in Fig. 3, the relative error between the estimates from qG-FDT for the linear operator and forcing of the SGS parameterization of a 40-EOF climate model and the à-posteriori result from the perturbed atmosphere itself. Also shown is the forcing error in applying the reduced qG-FDT (rqG-FDT) as explained in the main text.

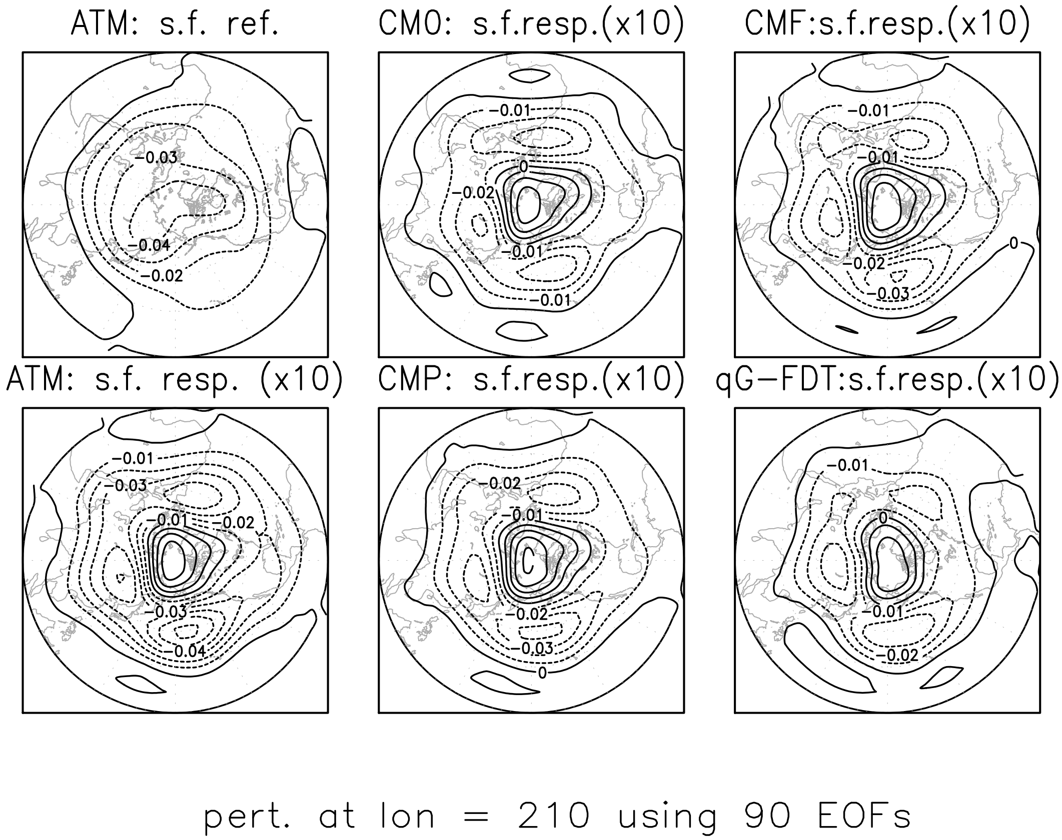

Fig. 5. The mean streamfunction of the toy atmosphere (upper left panel), its response to an anomalous vorticity forcing at longitude $\lambda_c = 210°$, projected onto 90 EOFs (lower left), the simulation of this response by a 90-EOF climate model with unmodified SGS parameterization (upper middle), by a climate model with SGS parameterization corrected à-posteriori by investigation of the perturbed atmosphere (lower middle), by a climate model with SGS parameterization corrected by rqG-FDT (upper right), and the direct estimation of the streamfunction response by qG-FDT (lower right). All values have been normalized by $a^2\Omega$, and the response has been multiplied by a factor 10.

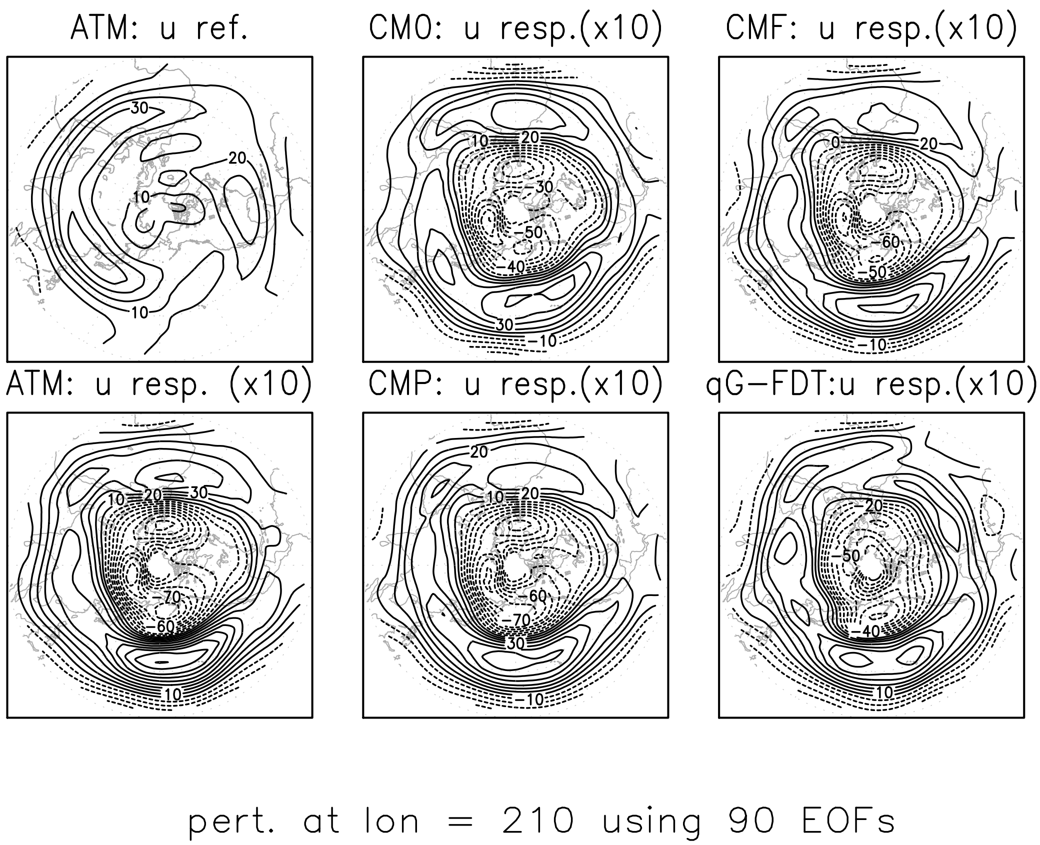

Fig. 6. As Fig. 5, but now for the zonal wind. Units are m/s.

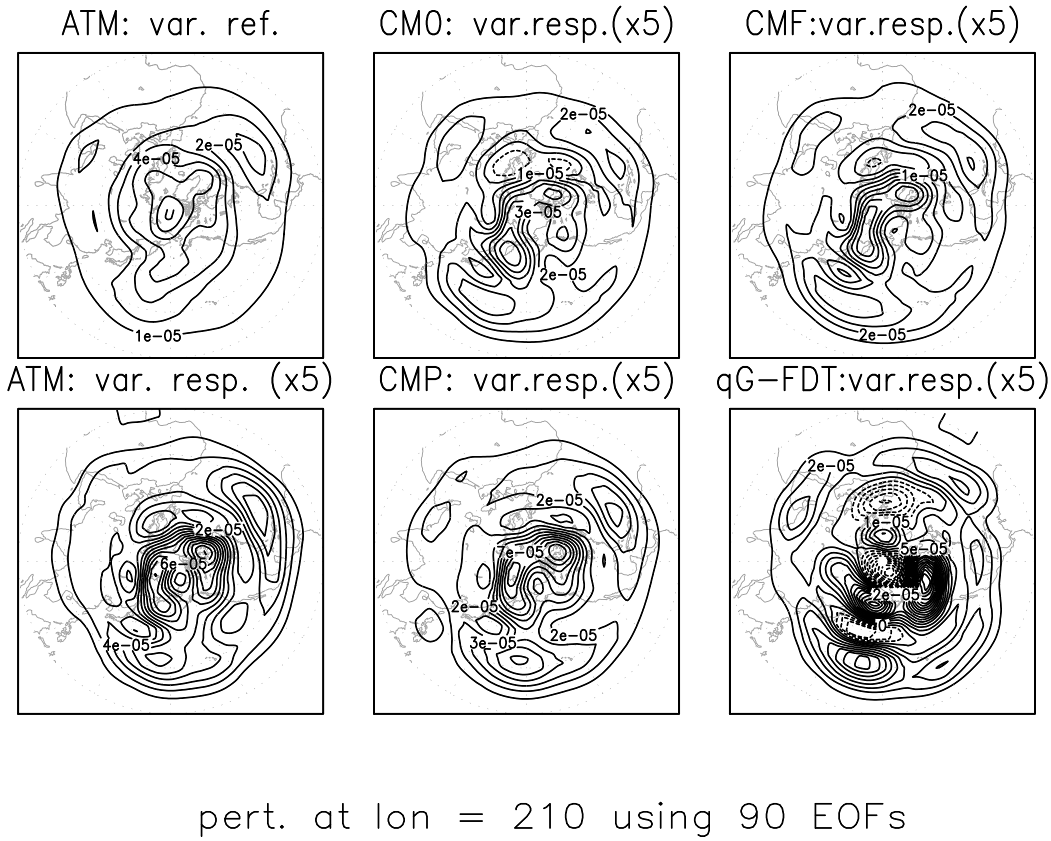

FIG. 7. As Fig. 5, but now for the streamfunction variance. All values have been normalized by $a^4\Omega^2$, and the response has been multiplied by a factor 5.

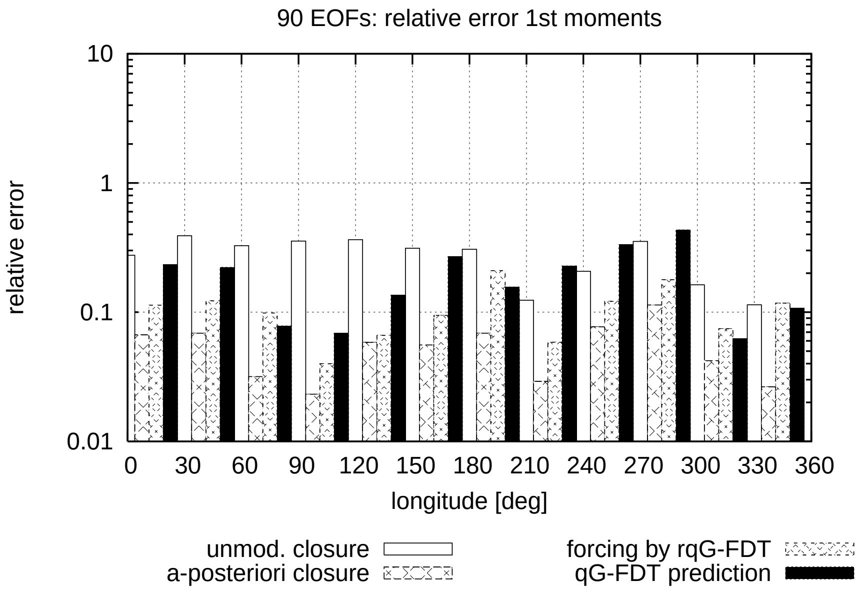

FIG. 8. Relative error in predicting the first-moment response of the toy atmosphere to anomalous local forcing at twelve different longitudes, and projected onto the leading 90 EOFs. Models used for the prediction are a 90-EOF low-order climate model with SGS parameterization obtained from unperturbed reference data (model CM0 in Figs. 5 - 7), a model with SGS parameterization adjusted à-posteriori on the basis of perterbed atmosphere data (model CMP in Figs. 5 - 7), a model using rqG-FDT for a before-hand prediction of the necessary change in the SGS parameterization (model CMF in Figs. 5 - 7), and direct application of qG-FDT.

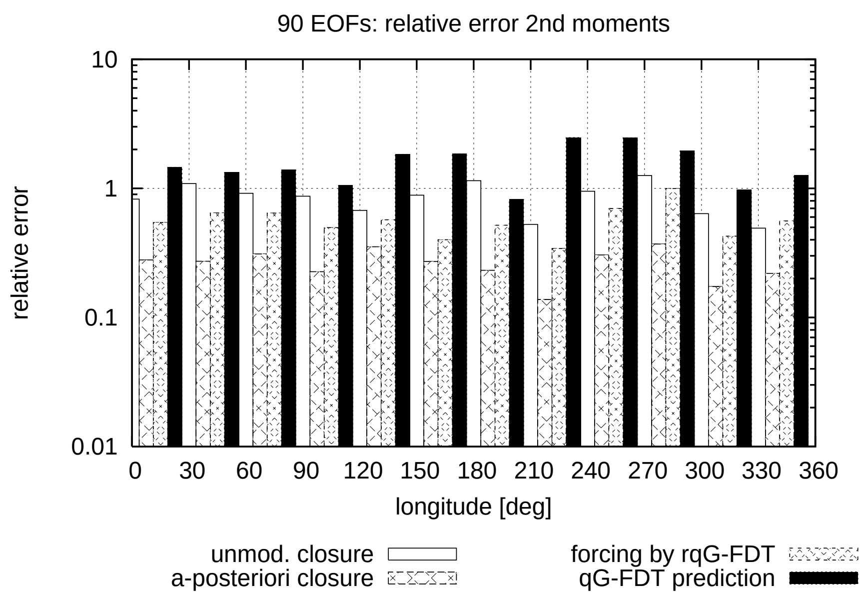

FIG. 9. As Fig. 8, but now the relative error in predicting the second-moment response of the toy atmosphere. Note the logarithmic scale.

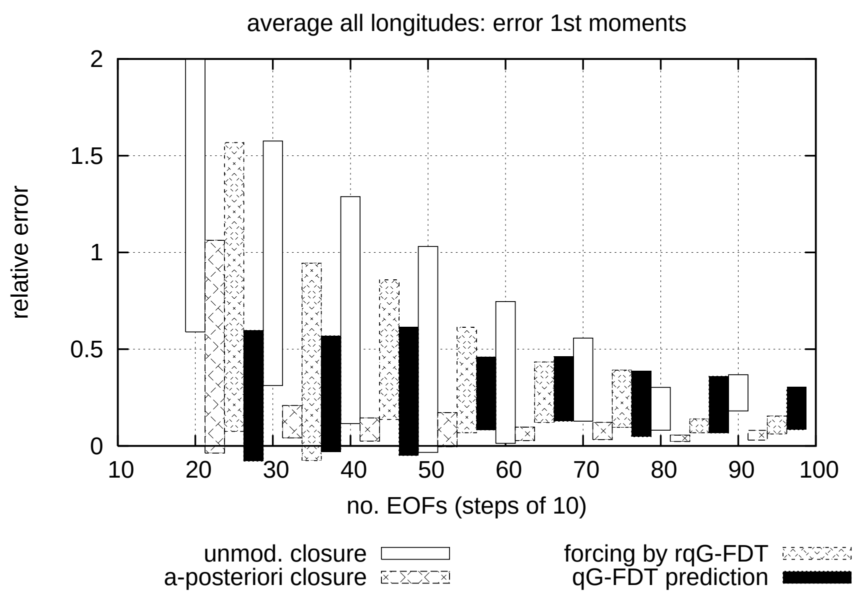

FIG. 10. For the same models as also analyzed in Fig. 8, now however at resolutions, i.e. number of basic EOFs, between 20 and 90, the mean first-moment error plus/minus the root-mean-square deviation, obtained from all twelve forcing cases, respectively.

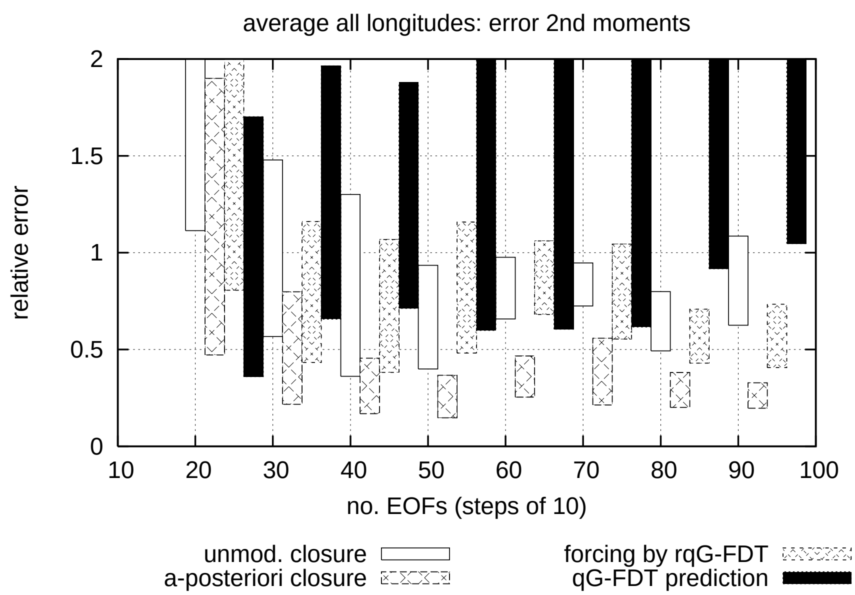

FIG. 11. As Fig. 10, but now for the second-moment errors.